Planck's Radiation law

Planck's law (colored curves) accurately described black body radiation and resolved the ultraviolet catastrophe (black curve).

Planck's law describes the electromagnetic radiation emitted by a black body in thermal equilibrium at a definite temperature. The law is named after Max Planck, who originally proposed it in 1900. It is a pioneering result of modern physics and quantum theory.

The spectral radiance of a body, B_ν, describes the amount of energy it gives off as radiation of different frequencies. It is measured in terms of the power emitted per unit area of the body, per unit solid angle that the radiation is measured over, per unit frequency. Planck showed that the spectral radiance of a body at absolute temperature T is given by

$$B_\nu(\nu, T) = \frac{2h\nu^3}{c^2} \frac{1}{e^{\frac{h\nu}{k_\mathrm{B}T}} - 1}$$

where k_B the Boltzmann constant, h the Planck constant, and c the speed of light in the medium, whether material or vacuum. The spectral radiance can also be measured per unit wavelength instead of per unit frequency. In this case, it is given by

$$B_\lambda(\lambda, T) = \frac{2hc^2}{\lambda^5} \frac{1}{e^{\frac{hc}{\lambda k_\mathrm{B}T}} - 1}.$$

The law may also be expressed in other terms, such as the number of photons emitted at a certain wavelength, or the energy density in a volume of radiation. The SI units of B_ν are W·sr^{-1}·m^{-2}·Hz^{-1}, while those of B_λ are W·sr^{-1}·m^{-3}.

In the limit of low frequencies (i.e. long wavelengths), Planck's law tends to the Rayleigh–Jeans law, while in the limit of high frequencies (i.e. small wavelengths) it tends to the Wien approximation.

Max Planck developed the law in 1900, originally with only empirically determined constants, and later showed that, expressed as an energy distribution; it is the unique stable distribution for radiation in thermodynamic equilibrium. As an energy distribution, it is one of a family of thermal equilibrium distributions which include the Bose–Einstein distribution, the Fermi–Dirac distribution and the Maxwell–Boltzmann distribution
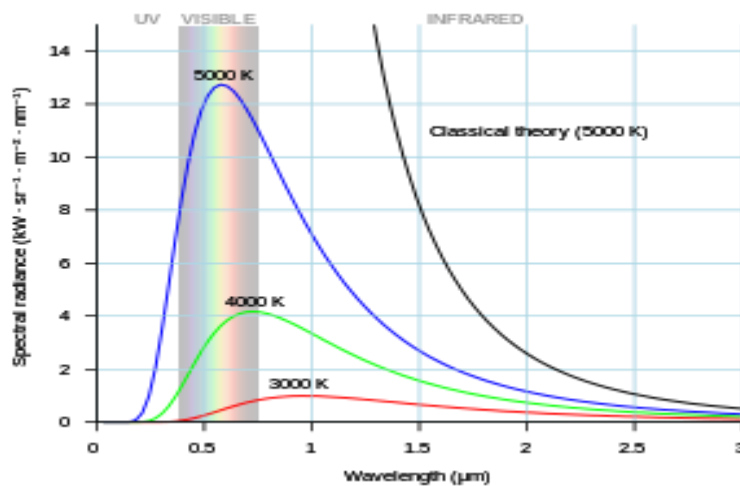
Introduction

Every physical body spontaneously and continuously emits electromagnetic radiation. Near thermodynamic equilibrium, the emitted radiation is nearly described by Planck's law. Because of its dependence on temperature, Planck radiation is said to be thermal radiation. The higher the temperature of a body the more radiation it emits at every wavelength. Planck radiation has a maximum intensity at a specific wavelength that depends on the temperature. For example, at room temperature (~300 K), a body emits thermal radiation that is mostly infrared and invisible. At higher temperatures the amount of infrared radiation increases and can be felt as heat, and the body glows visibly red. At even higher temperatures, a body is dazzlingly bright yellow or blue-white and emits significant amounts of short wavelength radiation, including ultraviolet and even x-rays. The surface of the sun (~6000 K) emits large amounts of both infrared and ultraviolet radiation; its emission is peaked in the visible spectrum.

Planck radiation is the greatest amount of radiation that any body at thermal equilibrium can emit from its surface, whatever its chemical composition or surface structure.[5] The passage of radiation across an interface between media can be

characterized by the emissivity of the interface (the ratio of the actual radiance to the theoretical Planck radiance), usually denoted by the symbol ε. It is in general dependent on chemical composition and physical structure, on temperature, on the wavelength, on the angle of passage, and on the polarization. The emissivity of a natural interface is always between $\varepsilon = 0$ and 1.

Different forms

Planck's law can be encountered in several forms depending on the conventions and preferences of different scientific fields. The various forms of the law for spectral radiance are summarized in the table below. Forms on the left are most often encountered in experimental fields, while those on the right are most often encountered in theoretical fields.

Planck's law expressed in terms of different spectral variables

with *h*		with *ℏ*	
variable	Distribution	variable	distribution
Frequency ν	$B_\nu(\nu, T) = \dfrac{2h\nu^3}{c^2} \dfrac{1}{e^{h\nu/(k_B T)} - 1}$	Angular frequency ω	$B_\omega(\omega, T) = \dfrac{\hbar\omega^3}{4\pi^3 c^2} \dfrac{1}{e^{\hbar\omega/(k_B T)} - 1}$
Wavelength λ	$B_\lambda(\lambda, T) = \dfrac{2hc^2}{\lambda^5} \dfrac{1}{e^{hc/(\lambda k_B T)} - 1}$	Angular wavelength y	$B_y(y, T) = \dfrac{\hbar c^2}{4\pi^3 y^5} \dfrac{1}{e^{hc/(y k_B T)} - 1}$

Wavenumber $\tilde{\nu}$	$B_{\tilde{\nu}}(\tilde{\nu}, T) = 2hc^2\tilde{\nu}^3 \dfrac{1}{e^{hc\tilde{\nu}/(k_\mathrm{B}T)} - 1}$	Angular wavenumber k	$B_k(k, T) = \dfrac{\hbar c^2 k^3}{4\pi^3} \dfrac{1}{e^{\hbar ck/(k_\mathrm{B}T)} - 1}$

These distributions represent the spectral radiance of blackbodies—the power emitted from the emitting surface, per unit projected area of emitting surface, per unit solid angle, per spectral unit (frequency, wavelength, wave number or their angular equivalents). Since the radiance is isotropic (i.e. independent of direction), the power emitted at an angle to the normal is proportional to the projected area, and therefore to the cosine of that angle as per Lambert's cosine law, and is unpolarized.

Spectral energy density form

Planck's law can also be written in terms of the spectral energy density (u) by multiplying B by $4\pi/c$:

$$u_i(T) = \frac{4\pi}{c} B_i(T).$$

These distributions have units of energy per volume per spectral unit.

First and second radiation constants

In the above variants of Planck's law, the *Wavelength* and *Wave number* variants use the terms $2hc^2$ and hc/k_B which comprise physical constants only. Consequently, these terms can be considered as physical constants themselves,[14] and are therefore referred to as the first radiation constant c_{1L} and the second radiation constant c_2 with

$$c_{1L} = 2hc^2$$

and

$$c_2 = hc/k_B$$

Using the radiation constants, the *Wavelength* variant of Planck's law can be simplified to

$$L(\lambda, T) = \frac{c_{1L}}{\lambda^5} \frac{1}{\exp\left(\frac{c_2}{\lambda T}\right) - 1}$$

and the *Wave number* variant can be simplified correspondingly.

L is used here instead of *B* because it is the SI symbol for *spectral radiance*. The *L* in c_{1L} refers to that. This reference is necessary because Planck's law can be reformulated to give spectral radiant existence *M* *(λ,T)* rather than *spectral radiance L(λ,T)*, in which case **c₁** replaces **c₁L**, with

$c_1 = 2\pi hc^2$

so that Planck's law for *spectral radiant existence* can be written as

$$M(\lambda, T) = \frac{c_1}{\lambda^5} \frac{1}{\exp\left(\frac{c_2}{\lambda T}\right) - 1}$$

Derivation

Gas in a box and Photon gas

Consider a cube of side *L* with conducting walls filled with electromagnetic radiation in thermal equilibrium at temperature T. If there is a small hole in one of the walls, the radiation emitted from the hole will be characteristic of a perfect black body. We will first calculate the spectral energy density within the cavity and then determine the spectral radiance of the emitted radiation.

At the walls of the cube, the parallel component of the electric field and the orthogonal component of the magnetic field must vanish. Analogous to the wave function of a particle in a box, one finds that the fields are superposition's of periodic functions. The three wavelengths λ_1, λ_2, and λ_3, in the three directions orthogonal to the walls can be:

$$\lambda_i = \frac{2L}{n_i},$$

Where the n_i are positive integers. For each set of integers n_i there are two linear independent solutions (modes). According to quantum theory, the energy levels of a mode are given by:

$$E_{n_1,n_2,n_3}(r) = \left(r + \frac{1}{2}\right) \frac{hc}{2L} \sqrt{n_1^2 + n_2^2 + n_3^2}. \qquad (1)$$

The quantum number r can be interpreted as the number of photons in the mode. The two modes for each set of n_i correspond to the two polarization states of the photon which has a spin of 1. Note that for $r = 0$ the energy of the mode is not zero. This vacuum energy of the electromagnetic field is responsible for the Casimir effect. In the following we will calculate the internal energy of the box at absolute temperature T.

According to statistical mechanics, the probability distribution over the energy levels of a particular mode is given by:

$$P_r = \frac{\exp(-\beta E(r))}{Z(\beta)}.$$

Here

$$\beta \stackrel{\text{def}}{=} 1/(k_\text{B}T).$$

The denominator $Z(\beta)$, is the partition function of a single mode and makes P_r properly normalized:

$$Z(\beta) = \sum_{r=0}^{\infty} e^{-\beta E(r)} = \frac{e^{-\beta\varepsilon/2}}{1 - e^{-\beta\varepsilon}}.$$

Here we have implicitly defined

$$\varepsilon \stackrel{\text{def}}{=} \frac{hc}{2L} \sqrt{n_1^2 + n_2^2 + n_3^2},$$

which is the energy of a single photon. As explained here, the average energy in a mode can be expressed in terms of the partition function:

$$\langle E \rangle = -\frac{d\log(Z)}{d\beta} = \frac{\varepsilon}{2} + \frac{\varepsilon}{e^{\beta\varepsilon} - 1}.$$

This formula, apart from the first vacuum energy term, is a special case of the general formula for particles obeying Bose–Einstein statistics. Since there is no restriction on the total number of photons, the chemical potential is zero.

If we measure the energy relative to the ground state, the total energy in the box follows by summing $\langle E \rangle - \frac{\varepsilon}{2}$ over all allowed single photon states. This can be done exactly in the thermodynamic limit as L approaches infinity. In this limit, ε becomes continuous and we can then integrate $\langle E \rangle - \frac{\varepsilon}{2}$ over the parameter. To calculate the energy in the box in this way, we need to evaluate how many photon states there are in a given energy range. If we write the total number of single photon states with energies between ε and $\varepsilon + d\varepsilon$ as $g(\varepsilon)d\varepsilon$, where $g(\varepsilon)$ is the density of states (which we'll evaluate in a moment), then we can write:

$$U = \int_0^\infty \frac{\varepsilon}{e^{\beta\varepsilon} - 1} g(\varepsilon)\, d\varepsilon. \qquad (2)$$

To calculate the density of states we rewrite equation (1) as follows:

$$\varepsilon \overset{\text{def}}{=} \frac{hc}{2L} n,$$

where n is the norm of the vector $\mathbf{n} = (n_1, n_2, n_3)$:

$$n = \sqrt{n_1^2 + n_2^2 + n_3^2}.$$

For every vector $\mathbf{n}$ with integer components larger than or equal to zero, there are two photon states. This means that the number of photon states in a certain region of n-space is twice the volume of that region. An energy range of $d\varepsilon$ corresponds to shell of thickness $dn = (2L/hc)d\varepsilon$ in n-space. Because the components of $\mathbf{n}$ have to be positive, this shell spans an octant of a sphere. The number of photon states $g(\varepsilon)d\varepsilon$, in an energy range $d\varepsilon$, is thus given by:

$$g(\varepsilon)\, d\varepsilon = 2\frac{1}{8} 4\pi n^2\, dn = \frac{8\pi L^3}{h^3 c^3} \varepsilon^2\, d\varepsilon.$$

Inserting this in Eq. (2) gives:

$$U = L^3 \frac{8\pi}{h^3 c^3} \int_0^\infty \frac{\varepsilon^3}{e^{\beta\varepsilon} - 1}\, d\varepsilon. \qquad (3)$$

From this equation one can derive the spectral energy density as a function of frequency $u_\nu(T)$ and as a function of wavelength $u_\lambda(T)$:

$$\frac{U}{L^3} = \int_0^\infty u_\nu(T)\, d\nu,$$

where:

$$u_\nu(T) = \frac{8\pi h \nu^3}{c^3} \frac{1}{e^{h\nu/k_\mathrm{B}T} - 1}.$$

And:

$$\frac{U}{L^3} = \int_0^\infty u_\lambda(T)\, d\lambda,$$

$$u_\lambda(T) = \frac{8\pi hc}{\lambda^5} \frac{1}{e^{hc/\lambda k_\mathrm{B}T} - 1}.$$

This is also a spectral energy density function with units of energy per unit wavelength per unit volume. Integrals of this type for Bose and Fermi gases can be expressed in terms of polylogarithms. In this case, however, it is possible to calculate the integral in closed form using only elementary functions. Substituting

$$\varepsilon = k_\mathrm{B}Tx,$$

in Eq. (3), makes the integration variable dimensionless giving:

$$u(T) = \frac{8\pi (k_\mathrm{B}T)^4}{(hc)^3} J,$$

where J is a Bose–Einstein integral given by:

$$J = \int_0^\infty \frac{x^3}{e^x - 1}\, dx = \frac{\pi^4}{15}.$$

The total electromagnetic energy inside the box is thus given by:

$$\frac{U}{V} = \frac{8\pi^5 (k_\mathrm{B}T)^4}{15(hc)^3},$$

where $V = L^3$ is the volume of the box.

The combination hc/k_B has the value 14 387.770 μm·K.

This is **not** the Stefan–Boltzmann law (which provides the total energy *radiated* by a black body per unit surface area per unit time), but it can be written more compactly using theStefan–Boltzmann constant σ, giving

$$\frac{U}{V} = \frac{4\sigma T^4}{c}.$$

The constant $4\sigma/c$ is sometimes called the radiation constant.

Since the radiation is the same in all directions, and propagates at the speed of light (*c*), the spectral radiance of radiation exiting the small hole is

$$B_\nu(T) = \frac{u_\nu(T)\,c}{4\pi},$$

which yields

$$B_\nu(T) = \frac{2h\nu^3}{c^2}\,\frac{1}{e^{h\nu/k_B T} - 1}.$$

It can be converted to an expression for $B_\lambda(T)$ in wavelength units by substituting ν by *c/λ* and evaluating

$$B_\lambda(T) = B_\nu(T)\left|\frac{d\nu}{d\lambda}\right|.$$