PHYSICS - Semester-I

Semester-I Unit-3rd.

PHY120C: PHYSICS: MECHANICS

Inverse Square Law Force:

All matter consist of fundamental particles, namely electrons, protons and neutrons. All what is going on in our universe is a consequence of the interaction between these fundamental particles (the term interaction means the mutual influence which fundamental particles exert on one another) there are four types of interaction between fundamental particles. These are.

(a) Gravitational force
(b) Electromagnetic force
(c) Nuclear force.
(d) Weak Nuclear force.

In Inverse Square force we define is that force whose magnitude is inversely proportional to the square of distance between the particles is called an inverse square force. The magnitude of electrostatic and gravitational force between two pt. particles at rest are given by

$$F = \frac{K}{r^2}.$$

Where. K - constant.

r - Distance between the centres of two pt. particles. Such forces are also called inverse Square Law.

Central forces:

In Case of gravitational interaction between two masses m_1 & m_2, the value of Constant

$$K = G m_1 m_2 \quad \text{—— (ii)}$$

∴ Gravitational interaction (force)

$$F = \frac{G m_1 m_2}{r^2}$$

Where $G = 6.67 \times 10^{-8}$ dyne cm^2/g^2 in cgs system.

and $G = 6.67 \times 10^{-11}$ Nm^2/Kg^2 in S.I System

In Case of Electrostatic interaction the force between two charges q_1 & q_2 the value of $K = \frac{1}{4\pi\epsilon_0} q_1 q_2$ —— (iii)

∴ Electrostatic force $F = \frac{1}{4\pi\epsilon_0} \frac{q_1 q_2}{r^2}$

Where $K = \frac{1}{4\pi\epsilon_0} = 9 \times 10^9 \; Nm^2/C^2$

& ϵ_0 is the permittivity of free space

Also the inverse Square Law of force is expressed in terms of inverse first power Law.

of potential energy (U)

i.e $F = -\frac{\partial U}{\partial r} = -\frac{K}{r^2}$

[assuming that forces of attraction is -ve sign is taken].

i.e $\frac{\partial U}{\partial r} = \frac{K}{r^2}$

or $U(r) = -\frac{K}{r} + \text{Constant } C$ —(iii)

When particles are at infinite distance from each other i.e when $r = \infty$ then $U(r) = 0$ and Eq (iii) gives that constant (C) is zero

$\therefore U(r) = -\frac{K}{r}$

The value of K is given by Eq's (i) & (ii) for gravitational & electrostatic forces in S.I system

$\therefore U(r) = \frac{G m_1 m_2}{r}$

and $U(r) = \frac{1}{4\pi\epsilon_0} \frac{q_1 q_2}{r}$

Central Forces: A force which is directed along the line joining the centres of the two interacting particles or bodies is called a central force. The gravitational and Electrostatic forces are examples of Central forces.

A central force can be written as

$$\vec{F} = F(r)\,\hat{e}_r$$

Where F(r) is the function of Distance and represents the magnitude of force. $\hat{e}_r$ is the unit vector and represents the direction of central force. It may be directed towards or away from origin.

CHARACTERISTICS OF Central Forces:

i) Central forces act along the line joining the centres of interacting particles or bodies.

ii) Central forces are Long-range forces and are effective even if the interacting bodies are very large distant apart.

iii) Central forces are Conservative in nature, i.e work done by these forces along a closed path is always zero.

iv) Central forces are derivable from scalar potential.

v) In central force motion, the angular momentum of the particle is conserved

vi) Central forces obey inverse sq. law

Reduction Of Two-Body Problem To Equivalent One-Body Problem.

Consider two bodies of masses m_1 & m_2 separated by a distance 'r' from each other. Let $\vec{F_1}$ & $\vec{F_2}$ be the central forces due to the mutual gravitational force of attraction between masses m_1 & m_2 respectively. The Equations of motion of two bodies are given by

$$m_1 \frac{d^2\vec{r_1}}{dt^2} = \vec{F_1} \quad \text{or} \quad \frac{d^2\vec{r_1}}{dt^2} = \frac{\vec{F_1}}{m_1} \quad -①$$

and $$m_2 \frac{d^2\vec{r_2}}{dt^2} = \vec{F_2} \quad \frac{d^2\vec{r_2}}{dt^2} = \frac{\vec{F_2}}{m_2} \quad -②$$

Subtracting ② from ①

$$\frac{d^2\vec{r_1}}{dt^2} - \frac{d^2\vec{r_2}}{dt^2} = \frac{F_1}{m_1} - \frac{F_2}{m_2}$$

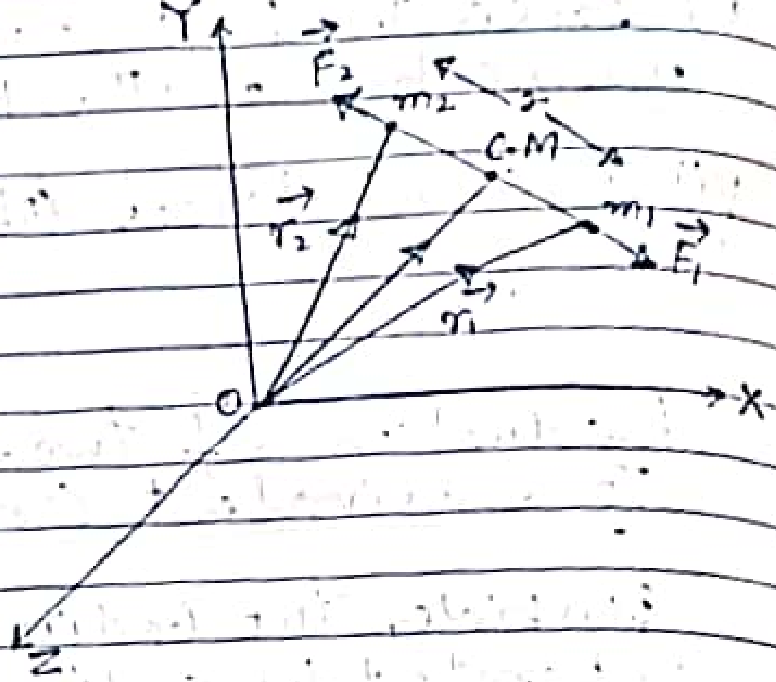
According to 3rd Law of motion

$$F_1 = -F_2 = F \text{(say)}$$

$$\therefore \frac{d^2\vec{r_1}}{dt^2} - \frac{d^2\vec{r_2}}{dt^2} = F/m_1 + F/m_2$$

⑥

or $\frac{d^2}{dt^2}(\vec{r}_1 - \vec{r}_2) = F/m_1 + F/m_2$ — (3)



But from figure $\vec{r} = \vec{r}_1 - \vec{r}_2$

$\therefore \frac{d^2\vec{r}}{dt^2} = \vec{F}/m_1 + \vec{F}/m_2$

or $\frac{d^2\vec{r}}{dt^2} = \vec{F}\left(\frac{1}{m_1} + \frac{1}{m_2}\right)$ — (4)

Take $\frac{1}{m_1} + \frac{1}{m_2} = \frac{1}{\mu}$

Where 'μ' is called Reduced Mass of System and is given by: $\mu = \frac{m_1 m_2}{m_1 + m_2}$

Eq. (4) Can be written as

$\frac{d^2\vec{r}}{dt^2} = F/\mu$ or $\vec{F} = \mu \frac{d^2\vec{r}}{dt^2}$ — (5)

According to Newton's Law of Gravitation

$\vec{F} = -G\frac{m_1 m_2}{r^2}\hat{e}_r$

Where $\hat{e}_r$ is the unit vector along the line joining the mass m_2 to m_1

-ve Sign indicates that gravitational force is attractive in nature

$\therefore$ Eq. (5) Can be written as

$$\mu \frac{d^2\vec{r}}{dt^2} = -\frac{G m_1 m_2}{r^2} \hat{e}_r \quad -(6)$$

Which is the equation of motion of one Body problem. Thus the relative motion is represented by the motion of a fictitious particle of mass μ acted on by the force F on the first particle.

EQUATION OF THE ORBIT IN A CENTRAL FORCE FIELD

To obtain the equation of the orbit of a particle of mass μ under central force field, we have to find the differential Equation in terms of r and θ

Consider the 2nd order differential equation of motion of a particle under the effect of a central force field

$$\mu \ddot{r} = \frac{L^2}{\mu r^3} + F(r)$$

$$\text{or} \quad \ddot{r} - \frac{L^2}{\mu^2 r^3} = \frac{F(r)}{\mu} \quad -(1)$$

$$\text{Put } r = \frac{1}{u} \quad -(2)$$

$$\text{then } \dot{r} = \frac{dr}{dt} = \frac{d}{dt}\left(\frac{1}{u}\right) = -\frac{1}{u^2}\frac{du}{dt}$$

$$-(3)$$

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$$\dot{r} = -\frac{1}{u^2}\frac{du}{d\theta}\cdot\frac{d\theta}{dt} = -\frac{1}{u^2}\dot{\theta}\frac{du}{d\theta}$$

$$\left[\because \frac{d\theta}{dt} = \dot{\theta}\right]$$

and $\ddot{r} = \frac{d}{dt}(\dot{r}) = \frac{d}{dt}\left[-\frac{1}{u^2}\dot{\theta}\frac{du}{d\theta}\right]$

$$= \frac{d}{d\theta}\left(-\frac{1}{u^2}\dot{\theta}\frac{du}{d\theta}\right)\frac{d\theta}{dt}$$

$$= \frac{d}{d\theta}\left(-\frac{1}{u^2}\dot{\theta}\frac{du}{d\theta}\right)\dot{\theta} \quad \text{— (4)}$$

But $\dot{\theta} = \frac{L}{\mu r^2} = \frac{L}{\mu}u^2$

$\therefore$ from Eq (3) $\dot{r} = -\frac{1}{u^2}\times\frac{L}{\mu}u^2\frac{du}{d\theta}$ — (5)

$$= -L/\mu\,\frac{du}{d\theta} \quad \text{— (5)}$$

and from Eq (4)

$$\ddot{r} = \left(-\frac{1}{u^2}\times\frac{Lu^2}{\mu}\frac{d^2u}{d\theta^2}\right)\times\frac{Lu^2}{\mu}$$

$$= -\frac{L^2u^2}{\mu^2}\frac{d^2u}{d\theta^2} \quad \text{— (6)}$$

Subs. the values of eq (ii) & (iii) and eq (vi) in Eq (i), we have.

$$-\frac{L^2u^2}{\mu^2}\frac{d^2u}{d\theta^2} - \frac{L^2u^3}{\mu^2} = \frac{F\left(\frac{1}{u}\right)}{\mu}$$

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or $\frac{L^2u^2}{\mu}\frac{d^2u}{d\theta^2}+\frac{L^2u^3}{\mu}=-F\left(\frac{1}{u}\right)$

or $\left[\frac{d^2u}{d\theta^2}+u\right]=-\frac{\mu}{L^2u^2}F\left(\frac{1}{u}\right)$ — (7)

This is the Differential Equation for the orbit of a particle under Central force field.

KEPLER'S LAWS OF PLANETARY MOTION

Johanne's Kepler analysed the data collected by the Danish astronomer Tycho-Brahe about the motion of stars and planets and concluded that the orbits of the planets were not circular as proposed by Copernicus but elliptical. He set forth his discoveries in the form of three Laws known as Kepler's Laws.

(1) First Law (Law of Orbits): Each planet moves around the Sun in an elliptical orbit with Sun at one of the foci of the ellipse.

We know that for bounded motion, the total energy of the system must be less than zero. In other words, the energy for bounded motion is negative. Hence, the particle must move in an elliptical orbit. Since motion of any planet around the Sun is bounded, so every particle (planet) around the Sun is bounded, thus every planet must move around the Sun in an elliptical orbit, which is Keplers' first Law.

The general equation of a Conic is given by.

$$\frac{1}{r} = l\left[1 + \epsilon \cos\theta\right].$$

$$\text{or } r = \frac{1}{l\left[1 + \epsilon \cos\theta\right]} \quad \text{--- (1)}$$

For elliptical orbit

When $\theta = 0$; $r = r_{min}$ = <u>Perihelion</u>

then from Eq (1)

$$r_{min} = \frac{1}{l(1+\epsilon)}$$

When $\theta = \pi$; $r = r_{max}$ = <u>Aphelion</u>

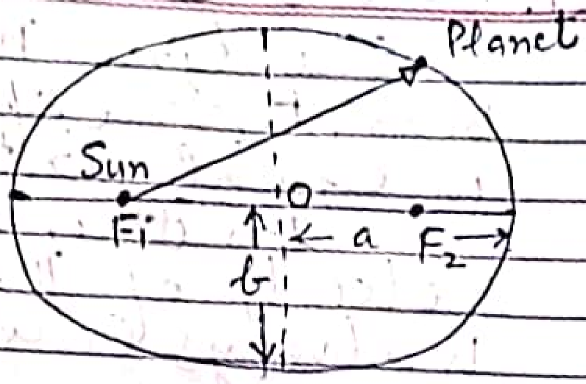
then from Eq (1)

$$r_{max} = \frac{1}{l(1-\epsilon)}$$

the Semi-major axis of the ellipse is given by.

(11) $a = \dfrac{r_{min} + r_{max}}{2}$

$$a = \frac{\frac{1}{l(1+\epsilon)} + \frac{1}{l(1-\epsilon)}}{2}$$



$$\text{or } a = \frac{1}{2l}\left[\frac{2}{1-\epsilon^2}\right]$$

$$= \frac{1}{l(1-\epsilon^2)}$$

Since $l = \dfrac{\mu K}{L^2}$ and $\epsilon = \left[1 + \dfrac{2EL^2}{\mu K^2}\right]^{1/2}$

$$\therefore a = \frac{L^2}{\mu K} \times \frac{1}{\left(1 - 1 - \frac{2EL^2}{\mu K^2}\right)}$$

$$\text{or } a = -\frac{K}{2E} \quad \text{or} \quad E = -\frac{K}{2a} \quad \text{— (2)}$$

This Equation Shows That the total energy of The planet moving in an elliptical orbit depends only on The length of major axis. Since total energy is negative, so motion of The planet around The Sun is bounded motion

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(2). SECOND LAW (Law of Areas)

The line joining the Sun and the planet sweeps out equal areas in equal intervals of time.

In other words, the areal velocity of the planet around the Sun is always constant.

Let position vector $\vec{r}$ rotates through an angle $d\theta$ in time dt, then area swept out by vector $\vec{r}$ in time dt is given by.

$$dA = \frac{1}{2} r (r\,d\theta) = \frac{1}{2} r^2 d\theta$$

$$\text{or} \quad \frac{dA}{dt} = \frac{1}{2} r^2 \frac{d\theta}{dt} = \frac{1}{2} r^2 \frac{L}{\mu r^2}$$

$$\text{or} \quad \boxed{\frac{dA}{dt} = \frac{L}{2\mu}} \quad \text{--- (1)} \qquad \left[\because \frac{d\theta}{dt} = \dot{\theta} = \frac{L}{\mu r^2}\right]$$

Since $\frac{L}{2\mu}$ = Constant, therefore

$$\frac{dA}{dt} \text{ (areal velocity)} = \text{Constant}$$

Which is Kepler's 2nd Law

3rd Law (Law of Periods)

"The Square of the Time period of a planet about the Sun is proportional to the cube of semi-major axis of the elliptical orbit"

i.e $$T^2 \propto a^3$$

The areal velocity of the planet is constant (according to Kepler's 2nd Law) and is given by.

$$\frac{dA}{dt} = \frac{L}{2\mu}$$

Let 'a' & 'b' be the semi-major and semi minor axis respectively of the ellipse, Therefore area of ellipse $= \pi ab$. The time period of planet is given by.

$$T = \frac{\text{Area of ellipse}}{\text{Areal velocity}} = \frac{\pi ab}{L/2\mu}$$

For an ellipse $b = a\sqrt{(1-e)^2}$

Since $$e = \left[1 + \frac{2EL^2}{\mu K^2}\right]^{1/2}$$

$$\therefore b = a\sqrt{\left(1 - 1 - \frac{2EL^2}{\mu K^2}\right)}$$

$$\text{or } b = a\sqrt{\frac{-2EL^2}{\mu K^2}}$$

But $E = -\dfrac{K}{2a}$

$$\therefore b = a\sqrt{\left(\frac{L^2}{\mu K a}\right)} = a^{1/2}\sqrt{\left(\frac{L^2}{\mu K}\right)}$$

$$\therefore T = \frac{\pi a \times a^{1/2}\sqrt{\left(\frac{L^2}{\mu K}\right)}}{\left(L/2\mu\right)} = \pi\left(a^{3/2}\right)\sqrt{\frac{4\mu}{K}}$$

Or $T^2 = \left(\dfrac{4\pi^2\mu}{K}\right)a^3$

Since $4\pi\mu/K$ is constant

Therefore $T^2 \propto a^3$

Which is Kepler's 3rd Law

ORBITS IN AN INVERSE SQUARE FORCE FIELD

The Shape of the Trajectory of the particle, depends upon the value of ϵ called eccentricity and ϵ also depends upon total energy E and angular momentum L.
It means that conic section may be Hyperbola; Parabola; Ellipse or Circle, depending upon the value of ϵ as follows.

1) If $\epsilon > 1$ or If $E > 0$ (+ve), the path is hyperbola.

2) If $\epsilon = 1$ or If $E = 0$, the path is Parabola

3) If $\epsilon < 1$ or If $E < 0$ (-ve), the path or Trajectory is ellipse.

4) If $\epsilon = 0$; The Conic is a Circle.

Thus the path of a particle under an inverse Square force field is either bounded (Circle or ellipse) or unbounded (Parabola or Hyperbola) depending on the value of ϵ as Shown.

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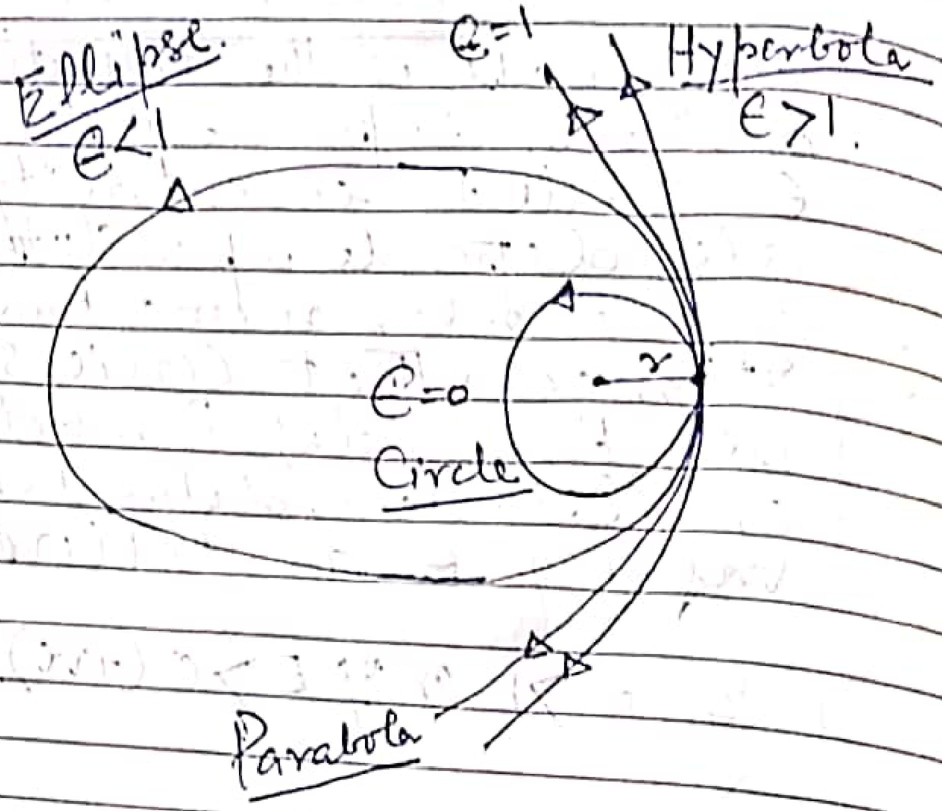


For E>0, The Trajectory of The body under influence of a repulsive force is always a Hyperbola the most familiar example is The path of a +vely charged α-particle moving under The influence of a positively charged nucleus. Thus for a repulsive force the Trajectory is always a hyperbola.

Gravitational field :-

Every particle of matter exerts a force of attraction on every other particle. This force is called the gravitational force of attraction. The area around about the attracting particle within which its gravitational force of attraction is perceptible is called gravitational field.

The intensity of gravitational field or the gravitational attraction at a pt. in a gravitational field is the force experienced by a unit mass placed at that point in the field. Thus gravitational attraction f_g at a point due to a particle of mass M at a distance r from the pt is in accordance with Newton's Law

$$f_g = -\frac{GM}{r^2}\hat{r}$$

Where $\hat{r} = \vec{r}/r$ is a unit vector along r. Thus force per unit mass is a measure of the field intensity.

Gravitational Potential

Now if a body B is moved in the gravitational field of another body A, a certain amount of work is to be done. If moved in the direction of the field, the work is done by the field itself and if moved against the field the work is done by some external agent

"The work done in moving a unit mass from infinity to any point in the gravitational field of a body is called the Gravitational Potential at that point due to the body". Gravitational potential at a pt. in a gravitational field may also be defined as the Potential energy of unit mass placed at that point. It is usually denoted by V_g and its value at a distance r from a body of mass M is given by:

$$V_g = -\int_{\infty}^{r} f_g \cdot dr = -\int_{\infty}^{r} \frac{GM}{r^2} \cdot \hat{r} = -\int_{\infty}^{r} \frac{GM}{r^2} dr$$

$$= \left[+\frac{GM}{r} \right]_{\infty}^{r} = -\frac{GM}{r}$$

It must be noted that Potential is a Scalar quantity while attraction is a vector Quantity.

Electrostatic field Potential:

The electric potential at a pt in an Electrostatic field is numerically equal to the workdone in bringing an infinitesimally small positive charge from infinity to that point.

i.e. Electrostatic field potential.

$$\phi = \lim_{q_0 \to 0} W/q_0$$

Where W – workdone in bringing charge q_0 from infinity to that point

S.I unit of Electric Potential = Joule/Coulomb.

Let E be the intensity of El. field i.e force experienced per unit +ve charge. Then the force experienced by a charge q placed in this field is given by

$$F_e = qE \quad \text{—— (1)}$$

∴ P.E of charge q is given by

$$V = -\int_{\infty}^{r} F_e \cdot dr = -\int_{\infty}^{r} qE \cdot dr \quad \text{—— (2)}$$

The Electrostatic Potential energy per unit charge is given by

$$\phi = V/q = -\int_{\infty}^{r} E \cdot dr \text{ i.e}$$

$$d\phi = -E \cdot dr \quad \text{—— (3)}$$

∴ Put $\phi' = \phi(x,y,z)$.

$$\therefore d\phi = \frac{\partial \phi}{\partial x}dx + \frac{\partial \phi}{\partial y}dy + \frac{\partial \phi}{\partial z}dz \quad — (4)$$

$$= \left(i\frac{\partial \phi}{\partial x} + j\frac{\partial \phi}{\partial y} + k\frac{\partial \phi}{\partial z}\right)\cdot(i\,dx + j\,dy$$

$$= (\text{grad}\,\phi)\cdot dr \quad — (4)$$

Comparing (3) & (4), we get

$$E = -\text{grad}\,\phi$$

i.e. Electric field Strength at any pt. may also be defined as the negative Gradient of potential at that pt.

Electric potential Difference: The electric potential difference between two pts. is the work done in moving an infinitesimally small +ve charge from one pt. to another.

If 'P' & 'Q' are two pts in an Electric field of intensity E, then Pot. Difference between P & Q is given by

$$\phi_Q - \phi_P = -\int_P^Q E\cdot dr$$

$$\text{Or}\quad \phi_P - \phi_Q = \int_P^Q E\cdot dr \quad — (5)$$

Electric Potential at a pt:

Consider a small charge q placed in an Electric field due to charge Q. Let the charge is placed at a distance r from charge Q. Then force on the charge q is given by:

$$F = \frac{1}{4\pi\epsilon_0}\frac{Qq}{r^2}\hat{r} \quad \text{— (6)}$$

$\hat{r}$ – unit vector along r

& $\epsilon_0 = 8.854 \times 10^{-12}\ C^2/N\text{-}m^2$

Then Potential energy of charge 'q' is given by.

$$V = -\int_{\infty}^{r} \vec{F}\cdot d\vec{r}$$

$$= -\int_{\infty}^{r} \frac{1}{4\pi\epsilon_0}\frac{Qq}{r^2}\hat{r}\cdot d\vec{r}$$

$$= -\int_{\infty}^{r} \frac{1}{4\pi\epsilon_0}\frac{Qq}{r^2}dr$$

$$= -\left[-\frac{1}{4\pi\epsilon_0}\frac{Qq}{r}\right]_{\infty}^{r}$$

$$\text{or. } V = \frac{1}{4\pi\epsilon_0}\frac{Qq}{r} \quad \text{— (7)}$$

This represents the Potential energy of charge q placed in Electric field due to charge Q at a distance r from it.

EQUIPOTENTIAL SURFACE:-

A surface, at all points of which the gravitational potential is the same, is called an equipotential surface.

Thus if we imagine a hollow sphere, of radius 'r' with a particle of mass 'm' at its centre, the potential at each point on it will be same viz, $-\frac{G.m}{r}$. The surface of the sphere is thus an equipotential surface. Now, since the difference of potential between any two pts. on an equipotential surface is zero, no work is done against the gravitational force in moving a unit mass along it. In other words there is no component of the gravitational field along an equipotential surface or the direction of field is at every point $\perp$ to it.

Consider two pts P & Q, a small distance δr apart on an equipotential surface AB. Let intensity of gravitational field at pt. P is I directed along PR at an angle θ with PQ

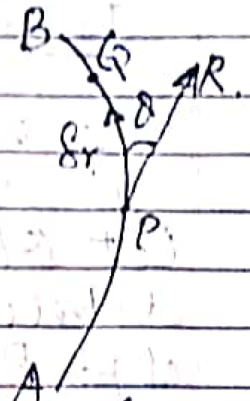
Component of field along PQ = $I\cos\theta$

Work done in moving unit mass from P to Q = $I\cos\theta . \delta r$

Since P & Q lie on equipotential surface

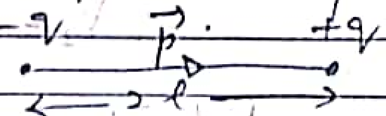
$\therefore I\cos\theta . \delta r = 0$

Since neither I nor δr is zero
we have $\cos\theta = 0$
or $\theta = 90°$
i.e. field is directed
along the Perpendicular
to the surface at pt. P



Thus direction of field
at every pt. on an equipotential
surface is Perpendicular to the surface
at that pt.

Electric Dipole :-
A system consisting of two equal & opposite charges separated by a fixed distance is called an Electric dipole
Consider $-q$ & $+q$
charges separated
by a distance $2l$
This system of charges is
called electric dipole.



Dipole moment is defined as the product of magnitude of any one charge & distance between the two charges. It is denoted by p.

$$\therefore p = q \times 2l$$

in vector form $\vec{p} = q \times 2\vec{l}$

The direction of dipole moment is from negative to positive charge
The line joining two charges is called its axial line whereas a line $\perp$ to the axial line is called equatorial line.

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Electric Potential at any point Due to an Electric dipole

Consider any pt. P at a distance 'r' from centre O of Electric dipole AB. Let OP makes an angle θ with vector dipole moment $\vec{p}$ & r_1 & r_2 be the distance of pt. P from $-q$ charge and $+q$ charge respectively.

Potential at P due to $-q$ Charge

$$V_1 = \frac{1}{4\pi\epsilon_0}\frac{-q}{r_1} = -\frac{1}{4\pi\epsilon_0}\frac{q}{r_1}$$

Potential at P due to charge $+q$

$$V_2 = \frac{1}{4\pi\epsilon_0}\frac{q}{r_2}$$

Potential at P due to dipole

$$V = V_1 + V_2 \quad \text{(Principle of Superposition)}$$

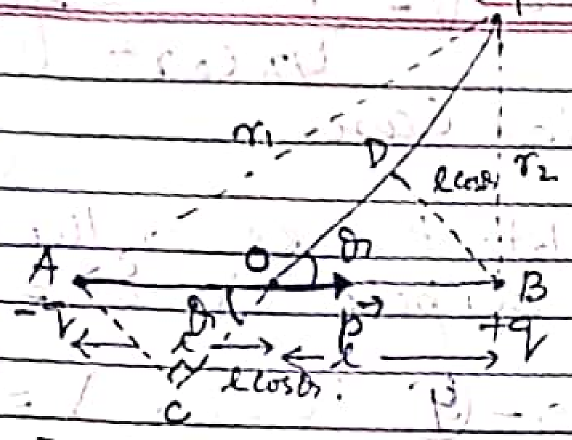
$$V = -\frac{1}{4\pi\epsilon_0}\frac{q}{r_1} + \frac{1}{4\pi\epsilon_0}\frac{q}{r_2}$$

$$\text{or } V = \frac{q}{4\pi\epsilon_0}\left[\frac{1}{r_2} - \frac{1}{r_1}\right] \quad -(1)$$

Draw a $\perp$ from A which meets the line OP at C when produced backward. Also draw BD $\perp$ on OP.

Then $r_1 = AP \sim CP = OP + OC = r + l\cos\theta$

and $r_2 = BP = DP = OP - OD = r - l\cos\theta$



Substituting the values of r_1 and r_2 in Eq ①

$$V = \frac{q}{4\pi\epsilon_0}\left(\frac{1}{(r - l\cos\theta)} - \frac{1}{(r + l\cos\theta)}\right)$$

$$= \frac{q}{4\pi\epsilon_0}\left[\frac{r + l\cos\theta - r + l\cos\theta}{r^2 - l^2\cos^2\theta}\right]$$

$$V = \frac{q}{4\pi\epsilon_0} \times \frac{2l\cos\theta}{r^2 - l^2\cos^2\theta}$$

$$= \frac{q\,2l\cos\theta}{4\pi\epsilon_0(r^2 - l^2\cos^2\theta)} \quad \text{--- ②}$$

i.e $$V = \frac{p\cos\theta}{4\pi\epsilon_0(r^2 - l^2\cos^2\theta)} \quad \text{--- ③} \quad \because p = q \cdot 2l$$

If $r >> l$, then we have from Eq ③

$$V = \frac{p\cos\theta}{4\pi\epsilon_0 r^2} \quad \text{--- ④}$$

Since $p\cos\theta = \vec{p} \cdot \hat{r}$ where $\hat{r}$ is a unit vector directed along OP

$$\therefore V = \frac{\vec{p}\cdot\hat{r}}{4\pi\epsilon_0 r^2} \quad \text{for } r >> l \quad \text{--- (5)}$$

Special Cases

1 If pt. P lies on the axial line of the dipole i.e. $\theta = 0°$

$\therefore$ Eq-(4) becomes $V = \frac{p}{4\pi\epsilon_0 r^2}$

or $V \propto \frac{1}{r^2}$ $\quad [\because \cos 0 = 1]$

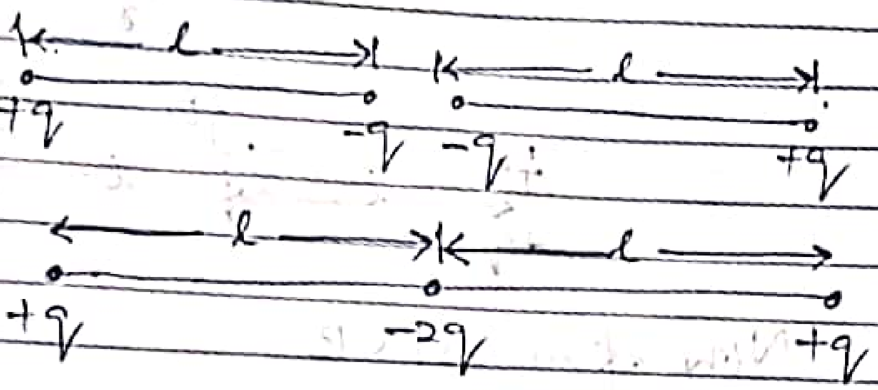
2 If pt. P lies on the equatorial line of the dipole i.e. $\theta = 90°$

Then $V = 0$ $\quad [\because \cos 90 = 0]$

Thus potential due to a dipole is zero at all points on the equatorial line of the dipole

Quadrupole & Quadrupole Moment

A quadrupole is a Combination of two identical electric dipoles separated by a fixed distance as shown.



Quadrupole moment is given by.

$$Q_d = 2q\,l^2$$

unit of quadrupole moment is Coulomb meter^2 (Cm^2)

Electric Potential Due to a Quadrupole

Consider a quadrupole ACD. let 'P' be the point whose coordinates are (r, θ) let r_1 & r_2 be the distances of charges +q at A and B respectively from pt. P

Electric Potential at P due to quadrupole is given by

$$V(r,\theta) = \frac{1}{4\pi\epsilon_0}\frac{q}{r_1} - \frac{1}{4\pi\epsilon_0}\frac{2q}{r} + \frac{1}{4\pi\epsilon_0}\frac{q}{r_2}$$

$$= \frac{q}{4\pi\epsilon_0}\left[\frac{1}{r_1} - 2/r + \frac{1}{r_2}\right] \quad \text{---(2)}$$

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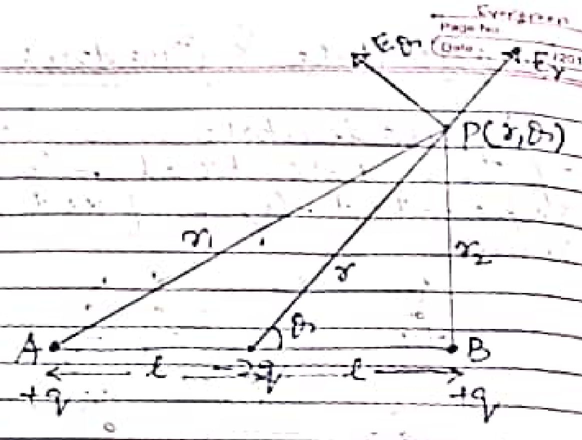

Now from ΔPCB

$$r_2^2 = r^2 + l^2 - 2rl\cos\theta$$

or $$r_2 = \left(r^2 + l^2 - 2rl\cos\theta\right)^{1/2}$$

$$= r\left[1 + (l/r)^2 - \frac{2l}{r}\cos\theta\right]^{1/2}$$

$$\frac{1}{r_2} = \frac{1}{r}\left[1 + \left\{(l/r) - \frac{2l}{r}\cos\theta\right\}\right]^{-1/2}$$

If $r >> l$, then using

$$(1-x)^{-1/2} = 1 - \tfrac{1}{2}x + \tfrac{3}{8}x^2 + \dots \text{ we have}$$

$$\frac{1}{r_2} = \frac{1}{r}\left[1 - \tfrac{1}{2}\left\{(l/r)^2 - \frac{2l}{r}\cos\theta\right\} + \left\{\tfrac{3}{8}(l/r)^2 - \frac{2l}{r}\cos\theta\right\}^2 + \dots\right]$$

$$= \frac{1}{r}\left[1 - \frac{l^2}{2r^2} + \frac{l}{r}\cos\theta + \tfrac{3}{8}\left\{\frac{l^2}{r^2} + \frac{4l^2}{r^2}\cos^2\theta - \frac{4l^3}{r^3}\cos\theta\right\} + \dots\right]$$

$$= \frac{1}{r}\left[1 - \frac{l^2}{2r^2} + \frac{l}{r}\cos\theta + \frac{3l^2}{8r^2} + \frac{3l^2}{2r^2}\cos^2\theta - \tfrac{3}{2}\frac{l^3}{r^3}\cos\theta \dots\right]$$

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Neglecting terms containing higher powers of (l/r), we have

$$\frac{1}{r_2} = \frac{1}{r}\left[1+\frac{l^2}{2r^2}(3\cos^2\theta-1)+\frac{l}{r}\cos\theta\right] \quad \text{---(ii)}$$

Similarly

$$\frac{1}{r_1} = \frac{1}{r}\left[1+\frac{l^2}{2r^2}(3\cos^2\theta-1)-\frac{l}{r}\cos\theta\right] \quad \text{---(iii)}$$

Putting values of (ii) + (iii) in Eq (1) we get:

$$V(r,\theta) = \frac{q}{4\pi\epsilon_0}\frac{1}{r}\left[1+\frac{l^2}{2r^2}(3\cos^2\theta-1)-\frac{l}{r}\cos\theta-2+1+\frac{l^2}{2r^2}(3\cos^2\theta-1)+\frac{l}{r}\cos\theta\right]$$

$$= \frac{q}{4\pi\epsilon_0}\cdot\frac{1}{r}\times\frac{l^2}{r^2}(3\cos^2\theta-1)$$

$$= \frac{1}{4\pi\epsilon_0}\cdot\frac{ql^2}{r^3}(3\cos^2\theta-1) \quad \text{---(iv)}$$

But $2ql^2 = Q_d$; quadrupole moment

$\therefore$ Eq (iv) can be written as

$$\boxed{V(r,\theta) = \frac{1}{4\pi\epsilon_0}\frac{Q_d}{2r^3}(3\cos^2\theta-1)} \quad \text{(A)}$$

Eq (A) gives the Potential Due to a Quadrupole.