or $lx + my + nz = l\alpha + m\beta + n\gamma$ ...(1)

[Type $lx + my + nz = p$]

The given conicoid is $ax^2 + by^2 + cz^2 = 1$ ...(2)

(1) will be a tangent to (2) if $\frac{l^2}{a} + \frac{m^2}{b} + \frac{n^2}{c} = p^2$

if $$\frac{l^2}{a} + \frac{m^2}{b} + \frac{n^2}{c} = (l\alpha + m\beta + n\gamma)^2$$ ...(3)

Now the equations of the perpendicular to (1) through (0, 0, 0) [*i.e.* Normal through (0, 0, 0)]

are $$\frac{x}{l} = \frac{y}{m} = \frac{z}{n}$$ ...(4)

To find the locus, we eliminate l, m, n from (3) and (4) :

$$\frac{x^2}{a} + \frac{y^2}{b} + \frac{z^2}{c} = (\alpha x + \beta y + \gamma z)^2$$

or $(\alpha x + \beta y + \gamma z)^2 = \frac{x^2}{a} + \frac{y^2}{b} + \frac{z^2}{c}$, which is the required result.

Example 7. *Obtain the tangent planes of the ellipsoid*

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1, \text{ which are parallel to } lx + my + nz = 0.$$

If 2r is the distance between the planes, show that a line through the origin and perpendicular to the planes lies on the cone

$$x^2 (a^2 - r^2) + y^2 (b^2 - r^2) + z^2 (c^2 - r^2) = 0.$$

Sol. Any plane parallel to

$lx + my + nz = 0$ is $lx + my + nz = K$ ...(1)

This touches the ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ ...(2)

if $a^2l^2 + b^2m^2 + c^2n^2 = p^2$

i.e. if $a^2l^2 + b^2m^2 + c^2n^2 = K^2$ *i.e.* if $K \pm \sqrt{a^2l^2 + b^2m^2 + c^2n^2}$.

Putting in (1), $lx + my + nz = \pm \sqrt{a^2l^2 + b^2m^2 + c^2n^2}$ ...(3)

which are the required equations.

Now $2r$= Distance between these two parallel planes

$$= \frac{2\sqrt{a^2l^2 + b^2m^2 + c^2n^2}}{\sqrt{l^2 + m^2 + n^2}} \quad \text{or} \quad r = \frac{\sqrt{a^2l^2 + b^2m^2 + c^2n^2}}{\sqrt{l^2 + m^2 + n^2}}$$

Squaring, $$r^2 = \frac{a^2l^2 + b^2m^2 + c^2n^2}{l^2 + m^2 + n^2}$$

or $$r^2 (l^2 + m^2 + n^2) = a^2l^2 + b^2 m^2 + c^2 n^2$$

or $(a^2 - r^2) l^2 + (b^2 - r^2) m^2 + (c^2 - r^2) n^2 = 0$ ...(4)

Now the equations of a line through (0, 0, 0) and $\perp$ to planes (3) are

$$\frac{x-0}{l} = \frac{y-0}{m} = \frac{z-0}{n} \text{ or } \frac{x}{l} = \frac{y}{m} = \frac{z}{n}$$ ...(5)

Eliminating l, m, n from (4) and (5), we have :

$x^2 (a^2 - r^2) + y^2 (b^2 - r^2) + z^2 (c^2 - r^2) = 0$, which is the required condition.

Example 11. *Prove that the locus of the foot of the perpendicular drawn from the centre of the ellipsoid* $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ *to any of its tangent planes is* $a^2x^2 + b^2y^2 + c^2z^2 = (x^2 + y^2 + z^2)^2$.

Sol. The given ellipsoid is $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ ...(1)

Let (x_1, y_1, z_1) be the point of contact.

$\therefore$ the tangent plane at (x_1, y_1, z_1) to (1) is

$$\frac{xx_1}{a^2} + \frac{yy_1}{b^2} + \frac{zz_1}{c^2} = 1 \quad ...(2)$$

where $$\frac{x_1^2}{a^2} + \frac{y_1^2}{b^2} + \frac{z_1^2}{c^2} = 1 \quad ...(3)$$

If (x', y', z') be the foot of $\perp$ from the centre (0, 0, 0) of (1) to (2), then the direction–cosines of this $\perp$ are proportional to

$x' - 0, y' - 0, z' - 0$ *i.e.* x', y', z'.

Since this $\perp$ is || to the normal to the plane (2),

$$\therefore \quad \frac{x'}{\frac{x_1}{a^2}} = \frac{y'}{\frac{y_1}{b^2}} = \frac{z'}{\frac{z_1}{c^2}} = K \text{ (say)}$$

$$\therefore \quad x_1 = \frac{a^2x'}{K}, y_1 = \frac{b^2y'}{K}, z_1 = \frac{c^2z'}{K} \quad \ldots(4)$$

Putting in (3), $\frac{a^2x'^2}{K^2} + \frac{b^2y'^2}{K^2} + \frac{c^2z'^2}{K^2} = 1$

or $K^2 = a^2x'^2 + b^2 y'^2 + c^2 z'^2 \quad \ldots(5)$

Since the point (x', y', z') lies on (2),

$$\therefore \quad \frac{x' x_1}{a^2} + \frac{y' y_1}{b^2} + \frac{z' z_1}{c^2} = 1 \quad \text{or} \quad \frac{x'}{a^2}\left(\frac{a^2x'}{K}\right) + \frac{y'}{b^2}\left(\frac{b^2y'}{K}\right) + \frac{z'}{c^2}\left(\frac{c^2z'}{K}\right) = 1$$

$$\text{or} \quad \frac{x'^2}{K} + \frac{y'^2}{K} + \frac{z'^2}{K} = 1 \quad \text{or} \quad x'^2 + y'^2 + z'^2 = K$$

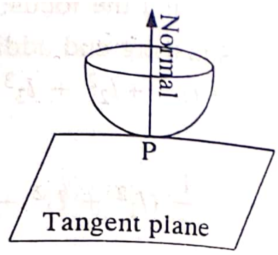
Squaring $(x'^2 + y'^2 + z'^2)^2 = K^2$

or $(x'^2 + y'^2 + z'^2)^2 = (a^2x'^2 + b^2y'^2 + c^2z'^2)$ [*Using (5)*]

Hence the locus of (x', y', z') is $a^2x^2 + b^2y^2 + c^2z^2 = (x^2 + y^2 + z^2)^2$, which is the required locus.

... $2y - z + 21 = 0$.

Art. 8. *(a)* **Normal. Def.** *The normal at any point of a surface is the straight line through the point of contact and perpendicular to the tangent plane at that point.*



***(b)* Equation of the normal.**

To find the equations of the normal at the point (x_1, y_1, z_1) of
***(i)* conicoid $ax^2 + by^2 + cz^2 = 1$.**

(ii) ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1.$

Sol. *(i)* The given conicoid is $ax^2 + by^2 + cz^2 = 1$ ...(1)

The equation of the tangent plane at (x_1, y_1, z_1) is $axx_1 + byy_1 + czz_1 = 1$

$\therefore$ the direction-cosines of the normal are proportional to ax_1, by_1, cz_1.

$\therefore$ the equations of the normal are

$$\frac{x - x_1}{ax_1} = \frac{y - y_1}{by_1} = \frac{z - z_1}{cz_1} \quad ...(3)$$

[**Notations :** If $f(x, y, z) = ax^2 + by^2 + cz^2 - 1 = 0$,

$$\therefore \frac{\partial f}{\partial x} = 2ax, \frac{\partial f}{\partial f} = 2by, \frac{\partial f}{\partial z} = 2cz \quad \therefore \quad \frac{\partial f}{\partial x_1} = 2ax_1, \frac{\partial y}{\partial y_1} = 2by_1, \frac{\partial f}{\partial z_1} = 2cz_1$$

$\therefore$ (3) can be written as : $\dfrac{x - x_1}{\frac{1}{2}\frac{\partial f}{\partial x_1}} = \dfrac{y - y_1}{\frac{1}{2}\frac{\partial f}{\partial x_1}} = \dfrac{z - z_1}{\frac{1}{2}\frac{\partial f}{\partial x_1}}$]

Actual Direction-cosines Form

If p be the length of $\perp$ from the centre (0, 0, 0) from the tangent plane (2),

then
$$p = \frac{1}{\sqrt{(ax_1)^2 + (by_1)^2 + (cz_1)^2}} \quad ...(4)$$

The direction-cosines of the normal are proportional to ax_1, by_1, cz_1

$\therefore$ the actual direction-cosines are

$$\frac{ax_1}{\sqrt{(ax_1)^2 + (by_1)^2 + (cz_1)^2}}, \frac{by_1}{\sqrt{(ax_1)^2 + (by_1)^2 + (cz_1)^2}}, \frac{cz_1}{\sqrt{(ax_1)^2 + (by_1)^2 + (cz_1)^2}}$$

i.e. ax_1p, by_1p, cx_1p [*Using (4)*]

$\therefore$ the equations of the normal at (x_1, y_1, z_1) are

$$\frac{x - x_1}{apx_1} = \frac{y - y_1}{bpy_1} = \frac{z - z_1}{cpz_1} \quad ...(5)$$

(ii) The given ellipsoid is $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ ...(1)

The equation of the tangent plane at (x_1, y_1, z_1) is

$$\frac{xx_1}{a^2} + \frac{yy_1}{b^2} + \frac{zz_1}{c^2} = 1 \quad ...(2)$$

$\therefore$ the direction-cosines of the normal are proportional to

$$\frac{x_1}{a^2}, \frac{y_1}{b^2}, \frac{z_1}{c^2}$$

$\therefore$ the equations of the normal are $\dfrac{x - x_1}{x_1/a^2} = \dfrac{y - y_1}{y_1/b^2} = \dfrac{z - z_1}{z_1/c^2}$ ...(3)

$c^4 \quad c^4 \quad c^4$

Art. 9. Number of Normals

To prove that there are six points on an ellipsoid the normals at which pass through a given point (α, β, γ). *(K.U. 2000 S)*

Proof. Let the given ellipsoid be $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ ...(1)

The normal at (x_1, y_1, z_1) is

$$\frac{x-x_1}{x_1/a^2}=\frac{y-y_1}{y_1/b^2}=\frac{z-z_1}{z_1/c^2}$$

This passes through (α, β, γ)

if $$\frac{\alpha-x_1}{x_1/a^2}=\frac{\beta-y_1}{y_1/b^2}=\frac{\gamma-z_1}{z_1/c^2}=\text{K (say)}$$ ...(2)

From first and last, $\alpha - x_1 = \frac{Kx_1}{a^2}$ or $\alpha = x_1 + \frac{Kx_1}{a^2} = \frac{x_1}{a^2}(a^2+K)$

$$x_1 = \frac{a^2\alpha}{a^2+K}$$

Similarly $$y_1 = \frac{b^2\beta}{b^2+K} \text{ and } z_1 = \frac{c^2\gamma}{c^2+K}$$ ...(3)

As $\frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}+\frac{z_1^2}{c^2}=1$ [∵ Point (x_1, y_1, z_1) lies on (1)]

$$\therefore \quad \frac{1}{a^2}\left(\frac{a^2\alpha}{a^2+K}\right)^2+\frac{1}{b^2}\left(\frac{b^2\beta}{b^2+K}\right)^2+\frac{1}{c^2}\left(\frac{c^2\gamma}{c^2+K}\right)^2=1$$

or $$\frac{a^2\alpha^2}{(a^2+K)^2}+\frac{b^2\beta^2}{(b^2+K)^2}+\frac{c^2\gamma^2}{(c^2+K)^2}=1$$

or $a^2\alpha^2 (b^2 + K)^2 (c^2 + K)^2 + b^2\beta^2 (c^2 + K)^2 (a^2 + K)^2 + c^2\gamma^2 (a^2 + K)^2 (b^2 + K)^2$
$= (a^2 + K)^2 (b^2 + K)^2 (c^2 + K)^2$

This equation, being of the sixth degree in K, gives six values of K.

Putting these values of K in (3), we get six points.

Hence the result.

Cor. Foot of the Normal.

From (3), $\left(\dfrac{a^2\alpha}{a^2 + K}, \dfrac{b^2\beta}{b^2 + K}, \dfrac{c^2\gamma}{c^2 + K}\right)$ are the co-ordinates of the foot of the normal.

Art. 10. *(a)* Cubic curve through the feet of normals

To prove that the six feet of the normals which can be drawn from a given point to an ellipsoid are the points of intersection of the ellipsoid and a certain cubic curve.

Proof. Let the given ellipsoid be $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} + \dfrac{z^2}{c^2} = 1$ ...(1)

Let the given point be (α, β, γ)

If (x_1, y_1, z_1) be the foot of the normal from (α, β, γ) to (1),

then $$x_1 = \frac{a^2\alpha}{a^2 + K}, y_1 = \frac{b^2\beta}{b^2 + K}, z_1 = \frac{c^2\gamma}{c^2 + K}$$

$\therefore$ the feet (x_1, y_1, z_1) of the normals lie on the curve

$$x = \frac{a^2\alpha}{a^2 + K}, y = \frac{b^2\beta}{b^2 + K}, z = \frac{c^2\gamma}{c^2 + K} \quad ...(2)$$

where K is a parameter having six values.

The points of intersection of the curve (2) with an arbitrary plane

$$ux + vy + wz + d = 0 \quad ...(3)$$

are given by

$$u.\frac{a^2\alpha}{a^2 + K} + v.\frac{b^2\beta}{b^2 + K} + w.\frac{c^2\gamma}{c^2 + K} + d = 0$$

or $ua^2\alpha (b^2 + K) (c^2 + K) + vb^2\beta (c^2 + K) (a^2 + K) + wc^2\gamma (a^2 + K) (b^2 + K)$
$+ d (a^2 + K) (b^2 + K) (c^2 + K) = 0$ ...(4)

This equation, being cubic in K, gives three values of K.

$\therefore$ the curve (2) meets the plane (3) in three points.

Thus the curve (2) is a cubic curve.

Hence the result.

***(b)* Quadric cone through six concurrent normals.**

To prove that six normals from a point to an ellipsoid lie on a cone of the second degree.

Proof. Let the given ellipsoid be $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} + \dfrac{z^2}{c^2} = 1$ ...(1)

Let any line through the given point (α, β, γ)

$$\frac{x - \alpha}{l} = \frac{y - \beta}{m} = \frac{z - \gamma}{n}$$ ...(2) be the normal

then $$l = \frac{px_1}{a^2}$$ [Art 8 (ii)]

$$= \frac{p}{a^2} \cdot \frac{a^2\alpha}{a^2 + K} \qquad \left[\because x_1 = \frac{a^2\alpha}{a^2 + K}\right]$$

$$\therefore \quad a^2 + K = \frac{p\alpha}{l} \qquad ...(3)$$

Similarly $$b^2 + K = \frac{p\beta}{m} \qquad ...(4)$$

and $$c^2 + K = \frac{p\gamma}{n} \qquad ...(5)$$

Multiplying (3) by $(b^2 - c^2)$, (4) by $(c^2 - a^2)$, (5) by $(a^2 - b^2)$ and adding, we get

$$a^2 (b^2 - c^2) + b^2 (c^2 - a^2) + c^2 (a^2 - b^2) + K [(b^2 - c^2) + (c^2 - a^2) + (a^2 - b^2)]$$

$$= \frac{p\alpha}{l}(b^2 - c^2) + \frac{p\beta}{m}(c^2 - a^2) + \frac{p\gamma}{n}(a^2 - b^2)$$

or $$0 + K[0] = \frac{p\alpha}{l}(b^2 - c^2) + \frac{p\beta}{m}(c^2 - a^2) + p\gamma(a^2 - b^2)$$

or $$\frac{\alpha}{l}(b^2 - c^2) + \frac{\beta}{m}(c^2 - a^2) + \frac{\gamma}{n}(a^2 - b^2) = 0 \qquad ...(6)$$

To find the locus, we eliminate l, m, n from (2) and (6)

$$\frac{\alpha}{x - \alpha}(b^2 - c^2) + \frac{\beta}{y - \beta}(c^2 - a^2) + \frac{\gamma}{z - \gamma}(a^2 - b^2) = 0$$

or $\alpha (b^2 - c^2)(y - \beta)(z - \gamma) + \beta (c^2 - a^2)(z - \gamma)(x - \alpha) + \gamma (a^2 - b^2)(x - \alpha)(y - \beta) = 0$

which is a cone of the second degree.

Hence the result.

ILLUSTRATIVE EXAMPLES

Example 6. *Prove that the feet of the six normals from* (α, β, γ) *to the ellipsoid* $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ *lie on the curve of intersection of the ellipsoid and the cone*

$$\frac{a^2(b^2 - c^2)\alpha}{x} + \frac{b^2(c^2 - a^2)\beta}{y} + \frac{c^2(a^2 - b^2)\gamma}{z} = 0.$$

Sol. The co-ordinates of the foot of the normal are given by

$$x = \frac{a^2\alpha}{a^2 + K}, \; y = \frac{b^2\beta}{b^2 + K}, \; z = \frac{c^2\gamma}{c^2 + K}$$

where K is a parameter having 6 values.

$$\therefore \quad a^2 + K = \frac{a^2\alpha}{x} \qquad ...(1)$$

Similarly $$b^2 + K = \frac{b^2\beta}{y} \qquad ...(2)$$

and $$c^2 + K = \frac{c^2\gamma}{z} \qquad \ldots(3)$$

Multiplying (1) by $(b^2 - c^2)$, (2) by $(c^2 - a^2)$, (3) by $(a^2 - b^2)$ and adding, we get

$$[a^2 (b^2 - c^2) + b^2 (c^2 - a^2) + c^2 (a^2 - b^2)] + K [(b^2 - c^2) + (c^2 - a^2) + (a^2 - b^2)]$$

$$= \frac{a^2(b^2 - c^2)\alpha}{x} + \frac{b^2(c^2 - \alpha^2)\beta}{y} + \frac{c^2(a^2 - b^2)\gamma}{z}$$

or $$\frac{a^2(b^2 - c^2)\alpha}{x} + \frac{b^2(c^2 - a^2)\beta}{y} + \frac{c^2(a^2 - b^2)\gamma}{z} = 0 \qquad \ldots(4)$$

Since the feet lie on the ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ ...(5)

Hence the feet of normals lie on the intersection of (4) and (5).

$\cdots p)$.

7.3. Polar plane.

To find the polar plane (use the definition given in Chapter 5) of a given point A (α, β, γ) with respect to the conicoid

$$ax^2 + by^2 + cz^2 = 1.$$ **(Rohilkhand 93)**

Suppose the equation of any line through $A\ (\alpha, \beta, \gamma)$ is

$$\frac{x-\alpha}{l} = \frac{y-\beta}{m} = \frac{z-\gamma}{n} = (r) \text{ say, } [l, m, n] \text{ being the d.cs.}$$

Any point on this line is $(lr + \alpha,\ mr + \beta,\ nr + \gamma)$.

If this line meets the given conicoid in this point, then we get

$$a\,(lr + \alpha)^2 + b\,(mr + \beta)^2 + c\,(nr + \gamma)^2 = 1$$

or $r^2(al^2+bm^2+cn^2)+2r\,(al\alpha+bm\beta+cn\gamma)+(a\alpha^2+b\beta^2+c\gamma^2-1) = 0.$

Since it is quadratic in r, the line intersects the conicoid at two points say P and Q. If r_1 and r_2 are the roots of this equation, then $AP = r_1$, $AQ = r_2$. Also

$$r_1 + r_2 = 2(al\alpha + bm\beta + cn\gamma)/(al^2 + bm^2 + cn^2),$$
$$r_1 . r_2 = (a\alpha^2 + b\beta^2 + c\gamma^2 - 1)/(al^2 + bm^2 + cn^2).$$

Now the polar plane is the locus of the point $R\ (x_1, y_1, z_1)$, where AP, AR and AQ are in $H.P.$, *i.e.*

$$\frac{2}{AR} = \frac{1}{AP} + \frac{1}{AQ} = \frac{AP + AQ}{AP\,.AQ}$$

or $$\frac{2}{AR} = \frac{r_1 + r_2}{r_1 . r_2} = \frac{-2(al\alpha + bm\beta + cn\gamma)}{(a\alpha^2 + b\beta^2 + c\gamma^2 - 1)}$$

If we put $AR = \rho$, then $$\frac{1}{\rho} = \frac{-(al\alpha + bm\beta + cn\gamma)}{(a\alpha^2 + b\beta^2 + c\gamma^2 - 1)}$$

or $$a\alpha l\rho + b\beta m\rho + c\gamma n\rho = -(a\alpha^2 + b\beta^2 + c\gamma^2 - 1) \qquad ...(i)$$

Now R is also a point on the line through $A\ (\alpha, \beta, \gamma)$, at a distance ρ from A. Hence the coordinates of R are also

$$(l\rho + \alpha,\ m\rho + \beta,\ n\rho + \gamma).$$

Therefore $x_1 = l\rho + \alpha,\ y_1 = m\rho + \beta,\ z_1 = n\rho + \gamma$

or $$x_1 - \alpha = l\rho,\ y_1 - \beta = m\rho,\ z_1 - \gamma = n\rho.$$

Putting these values of $l\rho$, $m\rho$, $n\rho$ in (i), we get

$$a\alpha\,(x_1 - \alpha) + b\beta\,(y_1 - \beta) + c\gamma\,(z_1 - \gamma) = -(a\alpha^2 + b\beta^2 + c\gamma^2 - 1)$$

or $$a\alpha x_1 + b\beta y_1 + c\gamma z_1 = 1.$$

Hence polar plane of (α, β, γ), which is the locus of (x_1, y_1, z_1), is given by $$\mathbf{a\alpha x + b\beta y + c\gamma z = 1.} \qquad \textbf{...(7.12)}$$

The point (α, β, γ) is called the pole of the polar plane and for a given polar plane it can be determined as in case of sphere.

Conjugate points and conjugate planes.

Consider two points (x_1, y_1, z_1) and (x_2, y_2, z_2) and the conicoid $ax^2 + by^2 + cz^2 = 1$. The two polar planes are

$$ax_1x + by_1y + cz_1z = 1 \ ...(i), \quad ax_2x + by_2y + cz_2z = 1 \qquad ...(ii)$$

Note that if plane (i) passes through the point (x_2, y_2, z_2), then plane (ii) will necessarily pass through the point (x_1, y_1, z_1). Such points are called **conjugate points** and planes are called **conjugate planes.**

Polar lines. *Two lines are said to be polar lines to each other if the polar plane of every point on the one line passes through the other line and vice-versa.*

Ex. 7.8. *Show that locus of straight lines drawn through a fixed point (α, β, γ) at right angles to their polars with respect to $ax^2+by^2+cz^2 = 1$, is*

$$\sum \frac{\alpha}{(x-\alpha)}\left[\frac{1}{b}-\frac{1}{c}\right]=0.$$

(Rohilkhand 91 ; Kanpur 82 ; Lucknow 80, 76 ; Meerut 89, 90)

Sol. Let the line through (α, β, γ) perpendicular to its polar line be

$$\frac{x-\alpha}{l}=\frac{y-\beta}{m}=\frac{z-\gamma}{n}. \quad ...(i)$$

Now first of all we shall find the polar line of this. Any point on this line is $(lr+\alpha, mr+\beta, nr+\gamma)$. The polar plane of this point is

$$ax(lr+\alpha)+by(mr+\beta)+cz(nr+\gamma)=1.$$

or $$(a\alpha x+b\beta y+c\gamma z-1)+r(alx+bmy+cnz)=0,$$

which always (for all values of r) passes through the line

$$\left.\begin{array}{r} a\alpha x+b\beta y+c\gamma z-1=0, \\ alx+bmy+cnz=0 \end{array}\right\} \quad ...(ii)$$

Thus the polar line of (i) is (ii) as the polar plane of any point on the line (i) passes through the line (ii) and it can be shown that the polar plane of any point on (ii) also passes through the line (i).

If L, M, N are the direction ratios of the line (ii) then they are given by

$$\frac{L}{bc(n\beta-m\gamma)}=\frac{M}{ca(\gamma l-\alpha n)}=\frac{N}{ab(\alpha m-\beta l)}$$

According to question, (i) is $\perp$ to (ii). Hence

$$lL+mM+nN=0.$$

or $$lbc(n\beta-m\gamma)+mca(\gamma l-\alpha n)+nab(\alpha m-\beta l)=0$$

or $$\alpha mna(b-c)+\beta nlb(c-a)+\gamma lmc(a-b)=0$$

or $$\sum \frac{\alpha}{l}\left[\frac{1}{c}-\frac{1}{b}\right]=0. \quad \text{(dividing by } lmnabc)$$

Hence the locus of (i) subject to condition (iii), is

$$\sum \frac{\alpha}{(x-\alpha)}\left[\frac{1}{c}-\frac{1}{b}\right]=0 \text{ or } \sum \frac{\alpha}{(x-\alpha)}\left[\frac{1}{b}-\frac{1}{c}\right]=0.$$

Exercise 7.3

1. Find the pole of the plane $lx+my+ny=p$, w.r.t. the conicoid

(i) $ax^2+by^2+cz^2=1.$ **(Meerut 83)**

(ii) $x^2/a^2+y^2/b^2+z^2/c^2=1$ **(Sri Venkt. U. 93)**

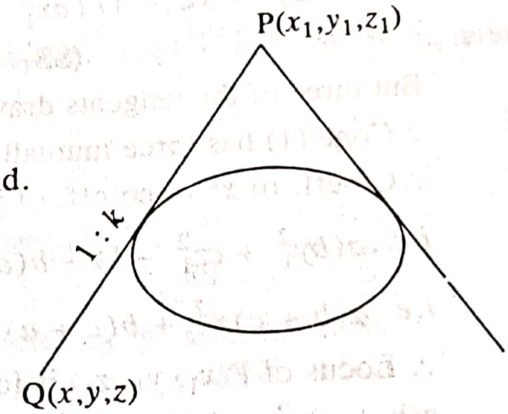
Art. 21. Enveloping Cone

To find the equation of the enveloping cone from the point (x_1, y_1, z_1) to the ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$.

Def. Enveloping Cone is the locus of the tangent lines drawn from a given point to a surface (here ellipsoid).

Sol. The equation of the ellipsoid is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$



Let P be the point (x_1, y_1, z_1).

Let $Q(x, y, z)$ be *any* point on a tangent line from P to be ellipsoid.

Any point on tangent PQ is (dividing PQ in the ratio $k : 1$).

$$\left(\frac{kx + x_1}{k+1}, \frac{ky + y_1}{k+1}, \frac{kz + z_1}{k+1}\right)$$

If it lies on ellipsoid (1), then

$$\frac{1}{a^2}\left(\frac{kx + x_1}{k+1}\right)^2 + \frac{1}{b^2}\left(\frac{ky + y_1}{k+1}\right)^2 + \frac{1}{c^2}\left(\frac{kz + z_1}{k+1}\right)^2 = 1$$

$$\frac{1}{a^2}(kx + x_1)^2 + \frac{1}{b^2}(ky + y_1)^2 + \frac{1}{c^2}(kz + z_1)^2 = (k+1)^2$$

$$k^2\left(\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} - 1\right) + 2k\left(\frac{xx_1}{a^2} + \frac{yy_1}{b^2} + \frac{zz_1}{c^2} - 1\right) + \left(\frac{x_1^2}{a^2} + \frac{y_1^2}{b^2} + \frac{z_1^2}{c^2} - 1\right) = 0 \quad \ldots(2)$$

ce the line PQ touches the conicoid (1)

).must have equal roots

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$\therefore$ Disc = 0 *i.e.*, $b^2 = 4ac$

$$\therefore 4\left(\frac{xx_1}{a^2} + \frac{yy_1}{b^2} + \frac{zz_1}{c^2} - 1\right)^2 = 4\left(\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} - 1\right)\left(\frac{x_1^2}{a^2} + \frac{y_1^2}{b^2} + \frac{z_1^2}{c^2} - 1\right)$$

or cancelling 4,

$$\left(\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} - 1\right)\left(\frac{x_1^2}{a^2} + \frac{y_1^2}{b^2} + \frac{z_1^2}{c^2} - 1\right) = \left(\frac{xx_1}{a^2} + \frac{yy_1}{b^2} + \frac{zz_1}{c^2} - 1\right)^2$$

which is the required equation of the enveloping cone.

Method to find the equation of Enveloping Cone

The enveloping cone is $SS_1 = T^2$

where $$S \equiv \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} - 1,\ S_1 \equiv \frac{x_1^2}{a^2} + \frac{y_1^2}{b^2} + \frac{z_1^2}{c^2} - 1$$

$$T \equiv \frac{xx_1}{a^2} + \frac{yy_1}{b^2} + \frac{zz_1}{c^2} - 1.$$

SOLVED EXAMPLES

Example 1. *Find the locus of points from which three mutually perpendicular tangents can be drawn to the surface $ax^2 + by^2 + cz^2 = 1$.*

Sol. Let $P(x_1, y_1, z_1)$ be the point whose locus is required. Equation of the enveloping cone of w.r.t. conicoid

$$ax^2 + by^2 + cz^2 = 1$$

(Each generator of this enveloping cone is a tangent from P to the conicoid) is

$$(ax^2 + by^2 + cz^2 - 1)\,(ax_1^2 - by_1^2 + cz_1^2 - 1) = (axx_1 + byy_1 + czz_1 - 1)^2$$

$$(SS_1 = T^2) \qquad \ldots(1)$$

But three of the tangents drawn from P are mutually $\perp$ (given).

$\therefore$ Cone (1) has three mutually $\perp$ generators.

$\therefore$ Co-eff. of x^2 + co-eff. of y^2 + co-eff. of $z^2 = 0$

i.e., $a(by_1^2 + cz_1^2 - 1) + b(ax_1^2 + cz_1^2 - 1) + c(ax_1^2 + by_1^2 - 1) = 0$

i.e., $a(b + c)x_1^2 + b(c + a)y_1^2 + c(a + b)z_1^2 = a + b + c$

$\therefore$ Locus of $P(x_1, y_1, z_1)$ is [changing (x_1, y_1, z_1) to (x, y, z)]

$a(b + c)x^2 + b(c + a)y^2 + c(a + b)z^2 = a + b + c.$

Art 22. Enveloping Cylinder

To find the equation of the enveloping cylinder of the conicoid $ax^2 + by^2 + cz^2 = 1$ whose generators are parallel to the line $\frac{x}{l} = \frac{y}{m} = \frac{z}{n}$

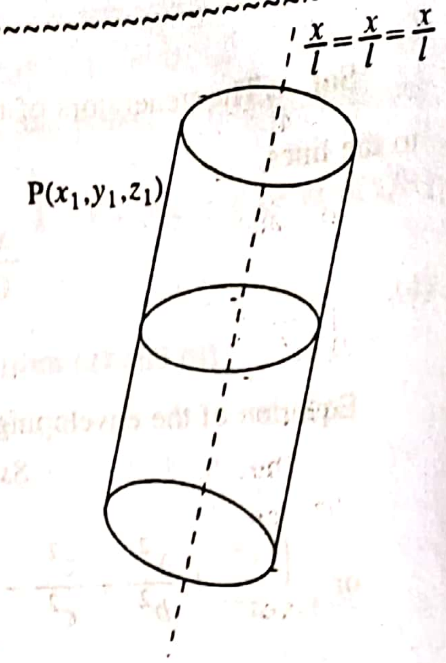
Sol. The Enveloping cylinder of the conicoid is the locus of the tangent lines to the conicoid drawn parallel to a given line (which is $\frac{x}{l} = \frac{y}{m} = \frac{z}{n}$)

Let $P(x_1, y_1, z_1)$ be ANY point on the cylinder.

∴ Equations of the generator of the enveloping cylinder through $P(x_1, y_1, z_1)$ parallel to the line

$$\frac{x}{l} = \frac{y}{m} = \frac{z}{n} \text{ are}$$

$$\frac{x - x_1}{l} = \frac{y - y_1}{m} = \frac{z - z_1}{n} \qquad ...(1)$$



∴ line (1) is a tangent line (being a generator of the enveloping cylinder) to the conicoid

$$ax^2 + by^2 + cz^2 = 1 \qquad ...(2)$$

Any point on line (1) is $(lr + x_1, mr + y_1, nr + z_1)$

If it lies on conicoid (2), then

$$a(lr + x_1)^2 + b(mr + y_1)^2 + c(nr + z_1)^2 = 1$$

or $r^2(al^2 + bm^2 + cn^2) + 2r(alx_1 + bmy_1 + cnz_1) + (ax_1^2 + by_1^2 + cz_1^2 - 1) = 0$ ...(3)

Since the line (1) touches the conicoid (2).

∴ (3) has equal roots

$$\therefore 4(alx_1 + bmy_1 + cnz_1)^2 - 4(al^2 + bm^2 + cn^2)(ax_1^2 + by_1^2 + cz_1^2 - 1) = 0$$

| Using $b^2 - 4ac = 0$ or cancelling 4,

$$(alx_1 + bmy_1 + cnz_1)^2 = (al^2 + bm^2 + cn^2)(ax_1^2 + by_1^2 + cz_1^2 - 1)$$

∴ Locus of (x_1, y_1, z_1) is

$$(ax^2 + by^2 + cz^2 - 1)(al^2 + bm^2 + cn^2) = (alx + bmy + cnz)^2 \qquad ...(4)$$

which is the required equation of the enveloping cylinder.

Method to find the equation of the enveloping cylinder. Let S denote the L.H.S. of the equation of the conicoid (after making the R.H.S. zero)

i.e., $S \equiv ax^2 + by^2 + cz^2 - 1$;

then $s_1 = al^2 + bm^2 + cn^2$

i.e., s_1 is obtained by putting l, m, n in S and omitting the constant term.

$$t = alx + bmy + cnz,$$

where t is the expression for the tangent plane at (l, m, n) after omitting the constant term.

∴ equation (4) of the enveloping cylinder can be put as

$$Ss_1 = t^2.$$

Example 2. *Prove that the enveloping cylinder of the ellipsoid*

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

whose generators are parallel to the lines

$$\frac{x}{0} = \frac{y}{\pm\sqrt{a^2 - b^2}} = \frac{z}{c}$$

meets the plane $z = 0$ *in a circle.*